# CMM3501
# Advanced Mathematical Methods

Systems of ODEs

# Outline

❖ **Fundamental matrix** solutions

❖ **Exponential matrices** + their properties

❖ Matrix exponential **solutions**; particular cases + general case

❖ Methods for computing matrix exponentials

❖ Applications to inhomogeneous systems of 1st order ODEs**; the method of variation of parameters**

❖ Particular case: **2nd order scalar ODEs** (the construction of a particular integral)

**Ciprian D. Coman**

$$\mathbf{x}' = \mathbf{Ax}$$

an $n \times n$ homogeneous linear system

$\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ are $n$ linearly independent

$$\mathbf{\Phi}(t) = \begin{bmatrix} | & | & & | \\ \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \\ | & | & & | \end{bmatrix} \longleftarrow \quad \textbf{fundamental matrix}$$

Ciprian D. Coman

# Fundamental matrix solutions

Because the column vector $\mathbf{x} = \mathbf{x}_j(t)$ of the fundamental matrix $\boldsymbol{\Phi}(t)$ in ▮ satisfies the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, it follows (from the definition of matrix multiplication) that the matrix $\mathbf{X} = \boldsymbol{\Phi}(t)$ itself satisfies the matrix differential equation $\mathbf{X}' = \mathbf{A}\mathbf{X}$. Because its column vectors are linearly independent, it also follows that the fundamental matrix $\boldsymbol{\Phi}(t)$ is nonsingular, and therefore has an inverse matrix $\boldsymbol{\Phi}(t)^{-1}$.

In terms of the fundamental matrix $\boldsymbol{\Phi}(t)$ in ▮, the general solution

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \cdots + c_n \mathbf{x}_n(t)$$

of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be written in the form

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{c}$$

where $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}^T$ is an arbitrary *constant* vector.

# Theorem 1

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## Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

➤ $$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

is given by

➤ $$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0.$$

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$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \text{with constant } n \times n \text{ coefficient matrix } \mathbf{A}$$

in the case where $\mathbf{A}$ has a complete set of $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ associated with the (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively.

$$\mathbf{x}_i(t) = \mathbf{v}_i e^{\lambda_i t} \qquad \text{for } i = 1, 2, \ldots, n$$

$$\mathbf{\Phi}(t) = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 e^{\lambda_1 t} & \mathbf{v}_2 e^{\lambda_2 t} & \cdots & \mathbf{v}_n e^{\lambda_n t} \\ | & | & & | \end{bmatrix} \qquad \text{is a fundamental matrix}$$

# Example 1

Find a fundamental matrix for the system

$$x' = 4x + 2y,$$
$$y' = 3x - y,$$

(13)

and then use it to find the solution of (13) that satisfies the initial conditions $x(0) = 1$, $y(0) = -1$.

The linearly independent solutions

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} 2e^{5t} \\ e^{5t} \end{bmatrix}$$

yield the fundamental matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

$$\mathbf{\Phi}(0) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \longrightarrow \mathbf{\Phi}(0)^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

# Example 1 (cont'd)

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$$\mathbf{x}(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \left(\tfrac{1}{7}\right) \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \left(\tfrac{1}{7}\right) \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$x(t) = \tfrac{3}{7}e^{-2t} + \tfrac{4}{7}e^{5t}, \quad y(t) = -\tfrac{9}{7}e^{-2t} + \tfrac{2}{7}e^{5t}.$$

**Remark** An advantage of the fundamental matrix approach is this: Once we know the fundamental matrix $\mathbf{\Phi}(t)$ and the inverse matrix $\mathbf{\Phi}(0)^{-1}$, we can calculate rapidly by matrix multiplication the solutions corresponding to different initial conditions.

$x(0) = 77, \ y(0) = 49$   (**new** initial conditions)

$$\mathbf{x}(t) = \tfrac{1}{7} \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 77 \\ 49 \end{bmatrix}$$

$$= \tfrac{1}{7} \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} -21 \\ 280 \end{bmatrix} = \begin{bmatrix} -3e^{-2t} + 80e^{5t} \\ 9e^{-2t} + 40e^{5t} \end{bmatrix}$$

# Exponential matrices

if $\mathbf{A}$ is an $n \times n$ matrix, then the **exponential matrix** $e^{\mathbf{A}}$ is the $n \times n$ matrix

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots + \frac{\mathbf{A}^n}{n!} + \cdots$$

The meaning of the infinite series on the RHS in the above definition:

$$\sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} = \lim_{k \to \infty} \left( \sum_{n=0}^{k} \frac{\mathbf{A}^n}{n!} \right)$$

where $\mathbf{A}^0 = \mathbf{I}$, $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AA}^2$, and so on; inductively, $\mathbf{A}^{n+1} = \mathbf{AA}^n$ if $n \geqq 0$

# Example 2

Consider the $2 \times 2$ diagonal matrix

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \longrightarrow \mathbf{A}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} a^2/2! & 0 \\ 0 & b^2/2! \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + a + a^2/2! + \cdots & 0 \\ 0 & 1 + b + b^2/2! + \cdots \end{bmatrix} \longrightarrow e^{\mathbf{A}} = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}$$

so the exponential of the *diagonal* $2 \times 2$ matrix $\mathbf{A}$ is obtained simply by exponentiating each diagonal element of $\mathbf{A}$.

The $n \times n$ analog of the $2 \times 2$ result in Example 2 is established in the same way. The exponential of the $n \times n$ diagonal matrix

$$\mathbf{D} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}$$

is the $n \times n$ diagonal matrix

$$e^{\mathbf{D}} = \begin{bmatrix} e^{a_1} & 0 & \cdots & 0 \\ 0 & e^{a_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{a_n} \end{bmatrix}$$

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▲ if **0** is the $n \times n$ zero matrix $\qquad e^{\mathbf{0}} = \mathbf{I}$

▲ $\mathbf{AB} = \mathbf{BA}$, then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$

▲ $\left(e^{\mathbf{A}}\right)^{-1} = e^{-\mathbf{A}}$

If $t$ is a scalar variable, then substitution of $\mathbf{A}t$ for $\mathbf{A}$ gives

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \cdots + \mathbf{A}^n \frac{t^n}{n!} + \cdots$$

# Example 3

$$A = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow e^{At} = ?$$

$$A^2 = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so } A^n = 0 \text{ for } n \geqq 3$$

$$e^{At} = I + At + \tfrac{1}{2}A^2 t^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} t + \tfrac{1}{2} \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} t^2 \longrightarrow e^{At} = \begin{bmatrix} 1 & 3t & 4t + 9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{bmatrix}$$

**Remark** If $A^n = 0$ for some positive integer $n$, then the exponential series in ▮ terminates after a finite number of terms, so the exponential matrix $e^A$ (or $e^{At}$) is readily calculated as in Example 3. Such a matrix—with a vanishing power—is said to be **nilpotent**.

# Example 4

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$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow e^{At} = ?$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = D + B$$

$$D = 2I$$

nilpotent matrix

$$e^{At} = e^{(D+B)t} = e^{Dt}e^{Bt} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 3t & 4t + 9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{2t} & 3te^{2t} & (4t + 9t^2)e^{2t} \\ 0 & e^{2t} & 6te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \mathbf{A} + \mathbf{A}^2 t + \mathbf{A}^3 \frac{t^2}{2!} + \cdots = \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \cdots\right)$$

$$\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \mathbf{A}e^{\mathbf{A}t}$$

Thus the matrix-valued function $\quad \mathbf{X}(t) = e^{\mathbf{A}t}$

satisfies the matrix differential equation $\quad \mathbf{X}' = \mathbf{A}\mathbf{X}$

Because the matrix $e^{\mathbf{A}t}$ is nonsingular, it follows that *the matrix exponential $e^{\mathbf{A}t}$ is a fundamental matrix for the linear system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$. In particular, it is *the* fundamental matrix $\mathbf{X}(t)$ such that $\mathbf{X}(0) = \mathbf{I}$. Therefore, Theorem 1 implies the following result.

# Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

➤ $$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

is given by

➤ $$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0,$$

and this solution is unique.

Thus the solution of homogeneous linear systems reduces to the task of computing exponential matrices. Conversely, if we already know a fundamental matrix $\mathbf{\Phi}(t)$ for the linear system $\mathbf{x}' = \mathbf{Ax}$, then the facts that $e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{C}$ $e^{\mathbf{A} \cdot 0} = e^{\mathbf{0}} = \mathbf{I}$ (the identity matrix) yield

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$$

So we can find the matrix exponential $e^{\mathbf{A}t}$ by solving the linear system $\mathbf{x}' = \mathbf{Ax}$

# Example 5

In Example 1 we found that the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

has fundamental matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \quad \text{with} \quad \mathbf{\Phi}(0)^{-1} = \tfrac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

Hence

$$e^{\mathbf{A}t} = \tfrac{1}{7}\begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \tfrac{1}{7}\begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

# Example 6

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Use an exponential matrix to solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 19 \\ 29 \\ 39 \end{bmatrix}$$

characteristic equation $(2 - \lambda)^3 = 0$

**TRIPLE EIGENVALUE!**

the eigenvector equation

$$(\mathbf{A} - 2\mathbf{I})\mathbf{v} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

the single solution $\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$

So we do NOT have the three linearly independent solutions associated with the triple eigenvalue $\lambda = 2$, which we would need to construct the fundamental matrix.

# Example 6 (cont'd)

However, the matrix $A$ is the same matrix that was used in Example 4.
We have already calculated the matrix exponential:

$$e^{At} = \begin{bmatrix} e^{2t} & 3te^{2t} & (4t + 9t^2)e^{2t} \\ 0 & e^{2t} & 6te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

Using Th. 2 we have

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) = \begin{bmatrix} e^{2t} & 3te^{2t} & (4t + 9t^2)e^{2t} \\ 0 & e^{2t} & 6te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 19 \\ 29 \\ 39 \end{bmatrix}$$

$$= \begin{bmatrix} (19 + 243t + 351t^2)e^{2t} \\ (29 + 234t)e^{2t} \\ 39e^{2t} \end{bmatrix}$$

The relatively easy calculation of $e^{At}$ carried out in Ex.4 was based on the observation that if

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

then $\mathbf{A} - 2\mathbf{I}$ is nilpotent:

$$(\mathbf{A} - 2\mathbf{I})^3 = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}$$

A similar result holds for any $3 \times 3$ matrix $\mathbf{A}$ having a triple eigenvalue $r$, in which case its characteristic equation reduces to $(\lambda - r)^3 = 0$. For such a matrix

$$(\mathbf{A} - r\mathbf{I})^3 = \mathbf{0}$$

the matrix $\mathbf{A} - r\mathbf{I}$ is nilpotent, and it follows that

$$e^{\mathbf{A}t} = e^{(r\mathbf{I}+\mathbf{A}-r\mathbf{I})t} = e^{r\mathbf{I}t} \cdot e^{(\mathbf{A}-r\mathbf{I})t} = e^{rt}\mathbf{I} \cdot \left[\mathbf{I} + (\mathbf{A}-r\mathbf{I})t + \tfrac{1}{2}(\mathbf{A}-r\mathbf{I})^2 t^2\right]$$

The calculations outlined above motivate a method of calculating $e^{\mathbf{A}t}$ for any $n \times n$ matrix whatsoever. Such a matrix has $n$ linearly independent generalised eigenvectors

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$$

Each generalized eigenvector $\mathbf{u}$ is associated with an eigenvalue $\lambda$ of $\mathbf{A}$ and has a *rank* $r \geqq 1$ such that

$$(\mathbf{A} - \lambda\mathbf{I})^r \mathbf{u} = \mathbf{0} \qquad \text{but} \qquad (\mathbf{A} - \lambda\mathbf{I})^{r-1}\mathbf{u} \neq \mathbf{0}$$

(If $r = 1$, then $\mathbf{u}$ is an ordinary eigenvector such that $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$.)

Even if we do not yet know $e^{\mathbf{A}t}$ explicitly, we can consider the function $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{u}$, which is a linear combination of the column vectors of $e^{\mathbf{A}t}$ and is therefore a solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{u}$. Indeed, we can calculate $\mathbf{x}$

explicitly in terms of $\mathbf{A}$, $\mathbf{u}$, $\lambda$, and $r$:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{u} = e^{(\lambda\mathbf{I} + \mathbf{A} - \lambda\mathbf{I})t}\mathbf{u} = e^{\lambda\mathbf{I}t}e^{(\mathbf{A} - \lambda\mathbf{I})t}\mathbf{u}$$

$$= e^{\lambda t}\mathbf{I}\left[\mathbf{I} + (\mathbf{A} - \lambda\mathbf{I})t + \cdots + (\mathbf{A} - \lambda\mathbf{I})^{r-1}\frac{t^{r-1}}{(r-1)!} + \cdots\right]\mathbf{u}$$

$$\mathbf{x}(t) = e^{\lambda t}\left[\mathbf{u} + (\mathbf{A} - \lambda\mathbf{I})\mathbf{u}t + (\mathbf{A} - \lambda\mathbf{I})^2\mathbf{u}\frac{t^2}{2!} + \cdots \right.$$

$$\left. + (\mathbf{A} - \lambda\mathbf{I})^{r-1}\mathbf{u}\frac{t^{r-1}}{(r-1)!}\right],$$

ASIDE:

$$e^{\lambda\mathbf{I}t} = e^{\lambda t}\mathbf{I}$$

# General Matrix Exponentials

If the linearly independent solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, ..., $\mathbf{x}_n(t)$ of $\mathbf{x}' = \mathbf{Ax}$ are calculated using ▮ with the linearly independent generalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n$, then the $n \times n$ matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{bmatrix} \quad ▮$$

is a fundamental matrix for the system $\mathbf{x}' = \mathbf{Ax}$. Finally, the specific fundamental matrix $\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$ satisfies the initial condition $\mathbf{X}(0) = \mathbf{I}$, and thus is the desired matrix exponential $e^{\mathbf{A}t}$. We have therefore outlined a proof of the following theorem.

## THEOREM 3  Computation of $E^{A\mathsf{T}}$

Let $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n$ be $n$ linearly independent generalized eigenvectors of the $n \times n$ matrix $\mathbf{A}$. For each $i$, $1 \leq i \leq n$, let $\mathbf{x}_i(t)$ be the solution of $\mathbf{x}' = \mathbf{Ax}$ given by ▮ substituting $\mathbf{u} = \mathbf{u}_i$ and the associated eigenvalue $\lambda$ and rank $r$ of the generalized eigenvector $\mathbf{u}_i$. If the fundamental matrix $\mathbf{\Phi}(t)$ is defined by ▮ then

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$$

We want to find a particular solution $\mathbf{x}_p$ of the nonhomogeneous linear system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

given that we have already found a general solution

$$\mathbf{x}_c(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t)$$

of the associated homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$

We first use the fundamental matrix $\mathbf{\Phi}(t)$ with column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ to rewrite the complementary function in ⬛ as

$$\mathbf{x}_c(t) = \mathbf{\Phi}(t)\mathbf{c}$$

where $\mathbf{c}$ denotes the column vector whose entries are the coefficients $c_1, c_2, \ldots, c_n$.

seek a particular solution of the form

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\mathbf{u}(t)$$

**AIM:** Find $\boldsymbol{u}(t)$ such that $\boldsymbol{x}_p(t)$ satisfies the original inhomogeneous ODE system.

$$\mathbf{x}'_p(t) = \mathbf{\Phi}'(t)\mathbf{u}(t) + \mathbf{\Phi}(t)\mathbf{u}'(t)$$

$$\mathbf{\Phi}'(t)\mathbf{u}(t) + \mathbf{\Phi}(t)\mathbf{u}'(t) = \mathbf{P}(t)\mathbf{\Phi}(t)\mathbf{u}(t) + \mathbf{f}(t)$$

$$\mathbf{\Phi}'(t) = \mathbf{P}(t)\mathbf{\Phi}(t)$$

$$\left. \right\} \longrightarrow \quad \mathbf{\Phi}(t)\mathbf{u}'(t) = \mathbf{f}(t)$$

$$\mathbf{u}'(t) = \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)$$

$$\boxed{\mathbf{u}(t) = \int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)\,dt}$$

## Variation of Parameters

If $\Phi(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on some interval where $\mathbf{P}(t)$ and $\mathbf{f}(t)$ are continuous, then a particular solution of the non-homogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

is given by

$$\mathbf{x}_p(t) = \Phi(t) \int \Phi(t)^{-1} \mathbf{f}(t)\, dt$$

This is the **variation of parameters formula** for first-order linear systems

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general solution
$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t) \int \mathbf{\Phi}(t)^{-1} \mathbf{f}(t)\, dt$$

The choice of the constant of integration in ▮ is immaterial, for we need only a single particular solution. In solving initial value problems it often is convenient to choose the constant of integration so that $\mathbf{x}_p(a) = \mathbf{0}$, and thus integrate from $a$ to $t$:

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t) \int_a^t \mathbf{\Phi}(s)^{-1} \mathbf{f}(s)\, ds$$

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(a) = \mathbf{x}_a$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(a)^{-1}\mathbf{x}_a + \mathbf{\Phi}(t) \int_a^t \mathbf{\Phi}(s)^{-1} \mathbf{f}(s)\, ds$$

$$\mathbf{x}_p(t) = e^{\mathbf{A}t} \int e^{-\mathbf{A}t} \mathbf{f}(t) \, dt$$

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}_0 + e^{\mathbf{A}t} \int_0^t e^{-\mathbf{A}t} \mathbf{f}(t) \, dt \qquad \mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

**OTHER WAYS TO WRITE THESE FORMULAE:**

$$\mathbf{x}_p(t) = \int e^{-\mathbf{A}(s-t)} \mathbf{f}(s) \, ds$$

$$\mathbf{x}(t) = e^{\mathbf{A}t} \mathbf{x}_0 + \int_0^t e^{-\mathbf{A}(s-t)} \mathbf{f}(s) \, ds$$

# Example 7

Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t e^{-2t}, \quad \mathbf{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Homogeneous system gives the fundamental matrix:

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \quad \text{with} \quad \mathbf{\Phi}(0)^{-1} = \tfrac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \cdot \tfrac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \tfrac{1}{7}\begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

# Example 7 (cont'd)

$$e^{-At}\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t e^{-As}\mathbf{f}(s)\,ds$$

$$= \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \int_0^t \tfrac{1}{7} \begin{bmatrix} e^{2s} + 6e^{-5s} & -2e^{2s} + 2e^{-5s} \\ -3e^{2s} + 3e^{-5s} & 6e^{2s} + e^{-5s} \end{bmatrix} \begin{bmatrix} -15se^{-2s} \\ -4se^{-2s} \end{bmatrix} ds$$

$$= \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \int_0^t \begin{bmatrix} -s - 14se^{-7s} \\ 3s - 7se^{-7s} \end{bmatrix} ds$$

$$= \begin{bmatrix} 7 \\ 3 \end{bmatrix} + \tfrac{1}{14} \begin{bmatrix} -4 - 7t^2 + 4e^{-7t} + 28te^{-7t} \\ -2 + 21t^2 + 2e^{-7t} + 14te^{-7t} \end{bmatrix}.$$

Therefore

$$e^{-At}\mathbf{x}(t) = \tfrac{1}{14} \begin{bmatrix} 94 - 7t^2 + 4e^{-7t} + 28te^{-7t} \\ 40 + 21t^2 + 2e^{-7t} + 14te^{-7t} \end{bmatrix}$$

# Example 7 (cont'd)

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$$\mathbf{x}(t) = \frac{1}{7} \begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix} \cdot \frac{1}{14} \begin{bmatrix} 94 - 7t^2 + 4e^{-7t} + 28te^{-7t} \\ 40 + 21t^2 + 2e^{-7t} + 14te^{-7t} \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} (6 + 28t - 7t^2)e^{-2t} + 92e^{5t} \\ (-4 + 14t + 21t^2)e^{-2t} + 46e^{5t} \end{bmatrix}.$$

# Variation of constants for 2<sup>nd</sup> order ODEs

$$y'' + Py' + Qy = f(t)$$

Write this ODE as a **system** of first-order ODEs:

$$y = x_1, \; y' = x_1' = x_2, \; y'' = x_1'' = x_2'$$

$$x_1' = x_2, \; x_2' = -Qx_1 - Px_2 + f(t)$$

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}, \quad \mathbf{P}(t) = \begin{bmatrix} 0 & 1 \\ -Q & -P \end{bmatrix}, \quad \text{and} \quad \mathbf{f}(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

# Variation of constants for 2nd order ODEs

$$\mathbf{x}_1 = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$$

linearly independent solutions of

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

The **Wronskian:**

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \longrightarrow \boldsymbol{\Phi}^{-1} = \frac{1}{W} \begin{vmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{vmatrix}$$

Therefore, by using the variation of constants formula $\quad \mathbf{x}_p = \boldsymbol{\Phi} \int \boldsymbol{\Phi}^{-1} \mathbf{f}\, dt$

$$\begin{bmatrix} y_p \\ y_p' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \int \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix} dt$$

$$= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \int \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix} dt.$$

The first component of this column vector is

$$y_p = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \int \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix} dt = -y_1 \int \frac{y_2 f}{W} dt + y_2 \int \frac{y_1 f}{W} dt$$

## Variation of Parameters

**(for second-order ODEs)**

If the nonhomogeneous equation $y'' + P(x)y' + Q(x)y = f(x)$ has complementary function $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, then a particular solution is given by

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} dx,$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions $y_1$ and $y_2$ of the associated homogeneous equation.

# Additional Notes